Lecture 2

01/22/2018

Review of Electrostatics (Cont'd)

We saw that for a uniform dipole layer with density D on a surface S;

$$\Phi(\vec{x}) = \frac{D}{4\pi\varepsilon_0}(\pm\Delta\Omega)$$

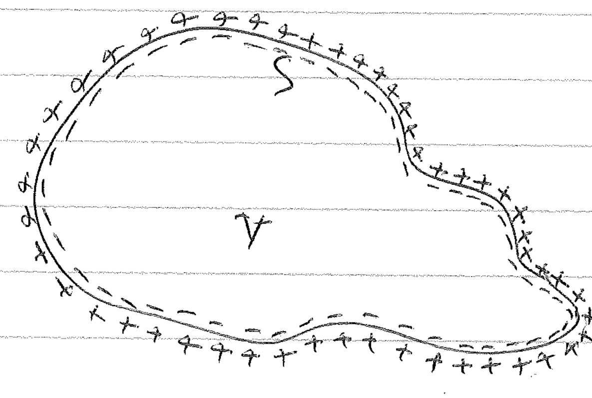
Here $\Delta\Omega$ is the total solid angle subtended by the surface S at the observation point. The positive sign arises when the observation point is above S, while the negative sign happens when the observation point is below S.

This results in discontinuity of the potential at the surface:

$$\Phi|_{S+} - \Phi|_{S-} = \frac{D}{\varepsilon_0}$$

Now, let us consider a closed dipole layer that has uniform dipole density D. In this case,

$$\Phi(\vec{x}) = \begin{cases} \frac{-D}{4\pi\varepsilon_0}4\pi = \frac{-D}{\varepsilon_0} & \text{if } \vec{x}\in V \\ 0 & \text{if } \vec{x}\notin V \end{cases}$$

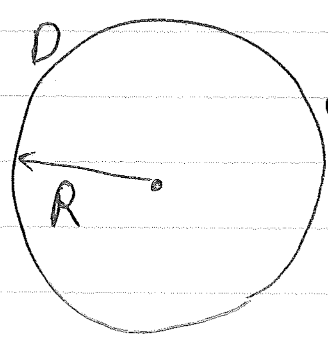


We note that if $\vec{x}$ is outside V, then the solid angle contributions exactly cancel out.

This result implies that the potential is constant both inside and outside of a closed dipole layer surface, hence $\vec{E}=0$ both inside and outside, but it is discontinuous at the surface thereby $|\vec{E}|$ infinitely large there.

An interesting consequence of this is that a combination of a charge layer and a dipole layer can be used to design field and potential configurations of interest. For example, a spherical shell with a uniform charge distribution and a uniform dipole-layer density can give rise to $\Phi=0$ both inside and at ∞ outside.

Inside:



Outside:

$$\Phi(\vec{x}) = -\frac{D}{\varepsilon_0} + \frac{R\sigma}{\varepsilon_0}$$

$$\boxed{\sigma = \frac{D}{R} \Rightarrow \Phi = 0}$$

$$\Phi(\vec{x}) = \frac{R^2\sigma}{\varepsilon_0 r}$$

Solution of Poisson Equation in a Volume V

Within a volume with specified non-trivial boundary contitions on Φ or $\frac{\partial \Phi}{\partial n}$, a different approach must be taken from the case in unbounded space. Let us start from the appropriate Green's function $G(\vec{x}, \vec{x}')$:

$$\nabla'^2 G(\vec{x}, \vec{x}') = -4\pi \delta^{(3)}(\vec{x} - \vec{x}')$$

We want to find the potential that satisfies the Poisson equation:

$$\nabla'^2 \Phi(\vec{x}') = \frac{-\rho(\vec{x}')}{\varepsilon_0}$$

From the two equations, we find:

$$\int_V \left[\Phi(\vec{x}') \nabla'^2 G(\vec{x}, \vec{x}') - G(\vec{x}, \vec{x}') \nabla'^2 \Phi(\vec{x}') \right] d\tau' = -4\pi \int_V \delta^{(3)}(\vec{x} - \vec{x}') \Phi(\vec{x}') d\tau'$$

$$+ \frac{1}{\varepsilon_0} \int_V \rho(\vec{x}') G(\vec{x}, \vec{x}') d\tau' \qquad (*)$$

However, recall that:

$$\Phi \nabla'^2 G - G \nabla'^2 \Phi = \nabla' \cdot (\Phi \nabla' G - G \nabla' \Phi)$$

Therefore, the left-hand side of * is equal to:

$$\oint_S \left[\Phi(\vec{x}') \frac{\partial G(\vec{x},\vec{x}')}{\partial n'} - G(\vec{x},\vec{x}') \frac{\partial \Phi(\vec{x}')}{\partial n'}\right] da'$$

We then find:

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \int_V \rho(\vec{x}') G(\vec{x},\vec{x}')\, d\tau' - \frac{1}{4\pi} \oint_S \left[\Phi(\vec{x}') \frac{\partial G}{\partial n'} - G(\vec{x},\vec{x}') \frac{\partial \Phi(\vec{x}')}{\partial n'}\right] da'$$

To solve the difficulty with the second term on the right-hand side of this expression, we choose G to obey the appropriated boundary condition:

(1) Dirichlet. In this case, we require that $G \equiv G_D(\vec{x},\vec{x}')|_S = 0$. This results in a unique solution:

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \int \rho(\vec{x}') G_D(\vec{x},\vec{x}')\, d\tau' - \frac{1}{4\pi} \oint \Phi(\vec{x}') \frac{\partial G_D(\vec{x},\vec{x}')}{\partial n'}\, da'$$

(2) Neumann. In this case, we choose $G \equiv G_N(\vec{x},\vec{x}')$ such that $\left.\frac{\partial G_N(\vec{x},\vec{x}')}{\partial n'}\right|_S = \frac{-4\pi}{S}$. Here S denotes the total surface area

Therefore, the left-hand side of * is equal to:

$$\oint_S \left[\Phi(\vec{x}') \frac{\partial G(\vec{x},\vec{x}')}{\partial n'} - G(\vec{x},\vec{x}') \frac{\partial \Phi(\vec{x}')}{\partial n'} \right] da'$$

We then find:

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \int_V \rho(\vec{x}') G(\vec{x},\vec{x}') \, d\tau' - \frac{1}{4\pi} \oint_S \left[\Phi(\vec{x}') \frac{\partial G}{\partial n'} - G(\vec{x},\vec{x}') \frac{\partial \Phi(\vec{x}')}{\partial n'} \right] da'$$

To solve the difficulty with the second term on the right-hand side of this expression, we choose G to obey the appropriated boundary condition:

(1) Dirichlet. In this case, we require that $G \equiv G_D(\vec{x},\vec{x}')|_S = 0$. This results in a unique solution:

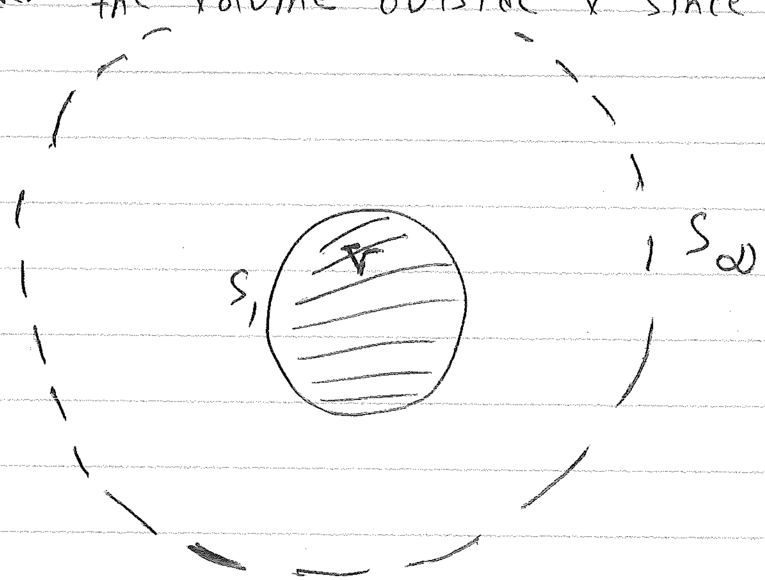
$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \int \rho(\vec{x}') G_D(\vec{x},\vec{x}') \, d\tau' - \frac{1}{4\pi} \oint \Phi(\vec{x}') \frac{\partial G_D(\vec{x},\vec{x}')}{\partial n'} da'$$

(2) Neumann. In this case, we choose $G \equiv G_N(\vec{x},\vec{x}')$ such that $\left.\frac{\partial G_N(\vec{x},\vec{x}')}{\partial n'}\right|_S = \frac{-4\pi}{S}$. Here S denotes the total surface area

of the boundary. Note that we cannot set $\frac{\partial G}{\partial n'}\Big|_S$ to zero because of the following:

$$\int_V \nabla'^2 G \, d\tau' = \int_V \vec{\nabla}' \cdot \vec{\nabla}' G \, d\tau' = \oint_S \frac{\partial G}{\partial n'} \, da' \overset{\text{from Poisson equation}}{=} -4\pi$$

For $\frac{\partial G}{\partial n'}\Big|_S = \text{Const.}$, we then find $\frac{\partial G}{\partial n'}\Big|_S = \frac{-4\pi}{S}$. This goes to zero when we consider the volume outside V since in this case $S = S_1 + S_\infty = \infty$:



As a result, in the Neumann case we find:

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \int_V \rho(\vec{x}') \, G_N(\vec{x}, \vec{x}') \, d\tau' + \frac{1}{4\pi} \int G_N(\vec{x}, \vec{x}') \frac{\partial \Phi(\vec{x}')}{\partial n'} \, da'$$

$$+ \underbrace{\frac{1}{S} \oint \Phi(\vec{x}') \, da'}_{\langle \Phi \rangle_S}$$

$\langle \Phi \rangle_S \rightarrow$ an additive constant up to which the Neumann problem

Some Interpretations and Notes

(1) G_D is the potential for the electrostatic problem in V that corresponds to a point charge $q = 4\pi\varepsilon_0$ at $\vec{x}$ with the boundary condition $G_D|_S = 0$. I.E., the potential due to a point charge inside a volume whose boundary is grounded.

(2) Finding the solution to the Poisson equation for arbitrary $\rho(\vec{x}')$ and boundary condition on Φ or $\frac{\partial \Phi}{\partial n}$ is reduced to finding the appropriate Green's function. For the Dirichlet problem, this is essentially a point charge problem with vanishing boundary condition.

(3) The Neumann Green's function is a slightly harder problem. However, as mentioned earlier, it is again a point charge problem with vanishing boundary condition in the case of the exterior problem.

(4) If S is a grounded conducting surface, the solution is:

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \int \rho(\vec{x}')\, G_D(\vec{x}, \vec{x}')\, d\tau'$$

We can write:

$$G_D(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|} + \tilde{G}_D(\vec{x},\vec{x}') \quad , \quad \nabla^2 \tilde{G}_D(\vec{x},\vec{x}') = 0$$

Thus:

$$\Phi(\vec{x}) = \underbrace{\frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}\, d\tau'}_{\text{Potential due to } \rho(\vec{x})} + \underbrace{\frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{x}')\, \tilde{G}_D(\vec{x}')\, d\tau'}_{\text{Potential due to charges induced on the surface}}$$

(5) The <u>method of images</u>, which we will discuss later, is one technique for finding the Dirichlet or Neumann Green's function for simple surfaces with symmetry. Another metheod is <u>separation of variables.</u> Once G_D or G_N is found, the problem of determining Φ everywhere inside V is solved by using the formulas given above.

(6) The Dirichlet Green's function is symmetric, i.e., $G_D(\vec{x},\vec{x}') = G_D(\vec{x}',\vec{x})$, while G_N can be so chosen (reciprocity theorem).